Supersymmetric time-continuous discrete random walks

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We apply the supersymmetric procedure to one-step random walks in one dimension at the level of the usual master equation, extending a study initiated by H.R. Jauslin [Phys. Rev. A **41**, 3407 (1990)]. A discussion of the supersymmetric technique for this discrete case is presented by introducing a formal second-order discrete master derivative and its "square root," and we solve completely, and in matrix form, the cases of homogeneous random walks (constant jumping rates). A simple generalization of Jauslin's results to two uncorrelated axes is also provided. There may be many applications, especially to bistable and multistable one-step processes.

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I. INTRODUCTION

A number of interesting results have been obtained in the past for the bistable Fokker-Planck equation [1] by means of the Witten supersymmetric approach [2]. Several years ago, in a paper hereafter denoted as I, Jauslin [3] developed the method of supersymmetric partners for time-continuous uncorrelated discrete random walks (RW's) in one dimension (1D). He discussed briefly the factorization of the discrete master operator, some of the properties of the superpartners, and as an application, the so called "addition of eigenvalues" method, generating bistability and multistability, in the supersymmetric framework. However, apparently Jauslin's treatment is not so directly linked to Witten's approach. To provide a more Witten-like picture to this discrete case was one of the motivations for our work. The paper is organized as follows. A discussion of the supersymmetric discrete master operator is presented in the next section, where we introduce a discrete master second derivative and its "square root" that enables us to write down a formal Riccati equation. In this way we come close to Witten's treatment allowing us to solve the simple case of homogeneous RW's. In Sec. III one can find an outline of Jauslin's "addition of eigenvalues" method. Then, in Sec. IV we make a simple generalization to two uncorrelated axes for homogeneous RW's, and we end up with some conclusions.

II. SUSY MASTER OPERATOR: FORMALISM

The 1D master equation used in I is the common one referring to three neighbor sites (or states),

$$-\frac{\partial P(n,t)}{\partial t} = -f(n-1)P(n-1,t) - b(n+1)P(n+1,t) + [f(n)+b(n)]P(n,t) = (M_-P)(n,t), \quad (1)$$

where $f(n-1)$ is the transition rate for one-step forward jumps starting at the $(n-1)$ site and $b(n+1)$ is the corresponding rate for the backward one-step jumps starting at the $(n+1)$ site. The jumps are random and the site (or state) is following a Markovian process with the evolution of the probability given by the master equation.

It is well known that the solution of the time-independent equation $M_-P_{st} = 0$ is [4]

$$P_{\rm st}(n) = \text{const} \times \prod_j \frac{f(j-1)}{b(j)}. \quad (2)$$

This form is a result of the detailed balancing condition $f(n)P_{\rm st}(n) = b(n+1)P_{\rm st}(n+1)$. The master operator M_- can be made Hermitian by defining a function

$$\psi(n,t) = [P_{\rm st}(n)]^{-1/2}P(n,t). \quad (3)$$

This function satisfies an operatorial equation of the type $-\frac{\partial\psi(n,t)}{\partial t} = (H_-\psi)(n,t)$, which is similar to Eq. (1),

$$-\frac{\partial\psi(n,t)}{\partial t} = -[f(n-1)b(n)]^{1/2}\psi(n-1,t) - [f(n)b(n+1)]^{1/2}\psi(n+1,t) + [f(n)+b(n)]\psi(n,t). \quad (4)$$

The "Hamiltonian" operator H_- is a symmetric positive operator with respect to the appropriate discrete l_2 scalar product. Also, for a normalizable $P_{\rm st}$ the lowest eigenvalue of H_- is nought and the ground state eigenfunction is $\phi_{{\rm gr},-} = [P_{\rm st}]^{1/2}$. The factorizing operators have been found in I to be

$$(A^+\psi)(n) = b^{1/2}(n+1)\psi(n+1) - f^{1/2}(n)\psi(n) \quad (5)$$

and

$$(A^-\psi)(n) = b^{1/2}(n)\psi(n-1) - f^{1/2}(n)\psi(n). \quad (6)$$

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These two factorizing operators can be written as

$$A^+ = b^{1/2}(n+1)\partial_n^+ + [b^{1/2}(n+1) - f^{1/2}(n)]$$
$$= b^{1/2}(n+1)\partial_n^+ + W^+(n) \qquad (7a)$$

or

$$A^+\psi(n) = b_{n+1}^{1/2}\left(\partial_n^+ + \frac{W_n^+}{b_{n+1}^{1/2}}\right)\psi(n) \qquad (7b)$$

and

$$A^- = -b^{1/2}(n)\partial_{n-1}^+ + [b^{1/2}(n) - f^{1/2}(n)]$$
$$= -b^{1/2}(n)\partial_{n-1}^+ + W^-(n) \qquad (8a)$$

or

$$A^-\psi(n) = b_n^{1/2}\left(-\partial_{n-1}^+ + \frac{W_n^-}{b_n^{1/2}}\right)\psi(n), \qquad (8b)$$

where $\partial_n^+\psi(n) = \psi(n+1) - \psi(n)$ and $\partial_{n-1}^+\psi(n) = \psi(n) - \psi(n-1)$ are discrete derivative operations, and the functions $W_-(n)$ and $W_+(n)$ correspond to the superpotential of the continuous limit as applied to $\psi(n)$. In this way, $H_- = A^+A^-$, and the superpartner will be $H_+ = A^-A^+$. Following I, we suppose the "Hamiltonian" H_+ to be of the same type as H_-. That means solutions of the type $b_+(n) = f_-(n)$ and $f_+(n) = b_-(n+1)$.

We write now the matrix form, i.e., the nilpotent master supercharges, of the factorizing operators, $Q_M^- = A_-\sigma_+ = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}$ and $Q_M^+ = A_+\sigma_- = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}$; $\sigma_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are Pauli matrices. In this realization, the matrix form of the "Hamiltonian" operator is

$$\mathcal{H} = \begin{pmatrix} A^+A^- & 0 \\ 0 & A^-A^+ \end{pmatrix} = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} \qquad (9)$$

defining the partner "Hamiltonians" as diagonal elements of the matrix one. They are partners in the sense that they are isospectral, apart from the ground state $\phi_{\mathrm{gr},-}$ of H_-, which is not included in the spectrum of H_+.

There is also one way of writing a formal discrete Witten scheme for the master operator. Firstly we write the operator action in the form

$$H_-\psi(n) = -(b_nf_n)^{1/2}\left\{\left[\left(\frac{b_{n+1}}{b_n}\right)^{1/2}\psi_{n+1} - 2\psi_n + \left(\frac{f_{n-1}}{f_n}\right)^{1/2}\psi_{n-1}\right] + \left[2 - \frac{b_n+f_n}{(b_nf_n)^{1/2}}\right]\psi_n\right\} \qquad (10)$$

and consider this operator as a one-site (local) operator, i.e., acting on $\psi(n)$. For this, one should introduce a discrete second derivative operation of the type

$$\partial_{M,n}^2\psi(n) = \left[\left(\frac{b_{n+1}}{b_n}\right)^{1/2}\psi_{n+1} - 2\psi_n + \left(\frac{f_{n-1}}{f_n}\right)^{1/2}\psi_{n-1}\right] \qquad (11)$$

that we call a *master discrete second derivative operation.* In this way the master equation can be written "locally" as follows:

$$H_-\psi(n) = (b_nf_n)^{\frac{1}{2}}\left[-\partial_{M,n}^2 + \left(\frac{b_n+f_n}{(b_nf_n)^{1/2}} - 2\right)\right]\psi(n). \qquad (12)$$

Then one can proceed formally with the Witten scheme, by defining the "square root" operator of $\partial_{M,n}^2$, or more exactly we have to go from the second master derivative to the first one by square-rooting, an operation to which we do not give here any rigorous meaning and we just denote it as $\sqrt{\partial_{M,n}^2}$. Then one may consider the symmetric discrete part $S(n) = \frac{1}{2}[\frac{b_n+f_n}{(b_nf_n)^{1/2}} - 2]$ as playing the role of the "Schrödinger" potential that we consider as a given quantity. Consequently

$$H_- = 2(b_nf_n)^{\frac{1}{2}}[\mathcal{A}^+\mathcal{A}^- + \epsilon], \qquad (13)$$

where we introduced in the usual way the symmetry breaking parameter ϵ (factorization energy), and

$$\mathcal{A}^\pm = \frac{1}{\sqrt{2}}[\pm\sqrt{\partial_{M,n}^2} + W_M(n)]. \qquad (14)$$

The formal Riccati equation for the master superpotential $W_M(n)$ would be

$$W_M^2(n) + \sqrt{\partial_{M,n}^2}W_M(n) = 2[S(n) - \epsilon]. \qquad (15)$$

The master superpotential $W_M(n)$ appears to be a factor acting on $\psi(n)$, but taking into account the nonlocal (three-site) character of the discrete master second derivative the above Riccati equation is formal as far as we introduced merely notations in order to put into evidence the similarity with the standard supersymmetric (susy) quantum mechanics.

An explicit case that can be solved completely is that of jump rates independent of the location along the axis, i.e., $b_n = c_1$, $f_n = c_2$, with c_1 and c_2 two positive constants. For free RW's $c_1 = c_2$, while for $c_1 \neq c_2$ we have an anisotropic RW. One can also normalize to unity, $c_1 + c_2 = c = 1$. In these cases there is a similarity between the master discrete second derivative and the popular discrete second derivative obtained when one starts from the continuum and makes the usual discretization up to $O(\Delta^2)$ terms,

$$f''(x) = \frac{1}{\Delta^2}[f(x+\Delta) - 2f(x) + f(x-\Delta)] + O(\Delta^2). \qquad (16)$$

Therefore in the continuum limit we identify the master second derivative with a common spatial second derivative implying the evolution equation

$$H_-\psi(n) = \left[-(c_1c_2)^{\frac{1}{2}}\frac{\partial^2}{\partial n^2} + [(c_1+c_2) - 2(c_1c_2)^{\frac{1}{2}}]\right]\psi(n) \qquad (17)$$

which being a normal Schrödinger equation is easily manipulated within the Witten susy scheme. Indeed, by the rescaling $x = 2^{-1/2}(c_1c_2)^{-1/4}n$ we get $H_-\psi(x) =$

$[-\frac{1}{2}\frac{\partial^2}{\partial x^2} + S(c_1, c_2)]\psi(x)$. This is a "Schrödinger-type" equation for a unit mass particle in the $S(c_1, c_2)$ potential. Next, in the framework of the addition of variables method (see the next section) we have to solve the equation $H_-\psi = -\lambda\psi$, with λ a positive constant, corresponding to the case $\epsilon = -\lambda$, less than the ground-state energy [5]. In the normalized to unity case, $c_1 + c_2 = 1$, the stationary solutions must be of the following hyperbolic type:

$$\psi_0(x) \propto \exp\left\{\pm\sqrt{2[1 - 2(c_1c_2)^{1/2} + \lambda]}x\right\}. \tag{18}$$

It is really easy to derive the exact form of ψ_0 if we require the ground-state wave function to be an acceptable one, that is, l_2 "integrable" and satisfying correct boundary conditions. The latter ones are that it must vanish at the end points $\pm\infty$, in agreement with the Sturm-Liouville theory (see I). To be more concrete, let us consider the free RW, $c_1 = c_2 = 1/2$. Then $S(1/2, 1/2) = 0$ and the corresponding susy quantum mechanical problem is that of a free particle. The Riccati master equation, Eq. (15), can be written

$$W^2(x) + \frac{\partial W}{\partial x} = 2\lambda \tag{19}$$

with the solution $W(x) = \sqrt{2\lambda}\tanh[\sqrt{2\lambda}(x - x_0)]$, where x_0 is an integration constant. Then in the unbroken susy case it is known that

$$\psi_0 = \exp\left[-\int W(x')dx'\right]. \tag{20}$$

By introducing the tanh solution one gets immediately $\psi_0 = \cosh(\sqrt{2\lambda}x)$, i.e., similar to the findings of Jauslin.

III. THE ADDITION OF EIGENVALUES

Jauslin's method of addition of eigenvalues is clearly exposed in I and we make here a brief outline. In fact, this method is a disguised form of a well-known procedure in susy quantum mechanics [5, 6], namely the factorization energy ϵ less than the ground-state energy. The idea consists in constructing a *shifted* "Hamiltonian" with respect to an initial one, H_0. Both Hamiltonians have zero energies for their ground states, but all the other eigenvalues of the shifted one are displaced with a chosen arbitrary distance, say λ_1, from the ground-state energy. Then, the proposal of Jauslin is to identify the shifted Hamiltonian with H_+. This condition leads to the following system of equations for the new jump rates f_1 and b_1:

$$f_1(n)b_1(n) = \alpha^2(n) = f_0(n)b_0(n-1), \tag{21}$$

$$f_1(n+1) + b_1(n) = \beta(n) = f_0(n) + b_0(n) + \lambda_1. \tag{22}$$

The following ansatz,

$$f_1(n) = \alpha(n)\frac{\phi_0(n)}{\phi_0(n-1)}, \tag{23a}$$

$$b_1(n) = \alpha(n)\frac{\phi_0(n-1)}{\phi_0(n)}, \tag{23b}$$

causes the geometrical mean equation, the first part of Eq. (21), to be identically satisfied, and moreover, Eq. (22) is turned into a Schrödinger-type equation for the initial Hamiltonian at the eigenvalue $-\lambda_1$, $H_0\phi_0 = -\lambda_1\phi_0$.

The real importance of the above method comes out when one is going to add even more eigenvalues (i.e., continuing the shifting) by an iteration procedure of the form

$$H_{k-1}\phi_{k-1} = -\lambda_k\phi_{k-1}\,, k \geq 1. \tag{24}$$

For the second eigenvalue, generating bistability in the stationary probability, one can obtain easily

$$f_2(n) = [f_1(n)b_1(n-1)]^{1/2}\frac{\phi_1(n)}{\phi_1(n-1)}, \tag{25a}$$

$$b_2(n) = [f_1(n)b_1(n-1)]^{1/2}\frac{\phi_1(n-1)}{\phi_1(n)}. \tag{25b}$$

The corresponding stationary probability, which is the quantity of interest, is

$$P_2^{st}(n) = \text{const} \times \frac{[f_1(n)b_1(n-1)]^{-1/2}}{\phi_1(n)\phi_1(n-1)}. \tag{26}$$

In Eq. (26), the jump rates f_1 and b_1 are determined through Eqs. (23) above, while the function ϕ_1 is given by

$$\phi_1(n) = b_1^{1/2}\bar{\phi}_0(n; \lambda_1 + \lambda_2) - f_1^{1/2}(n)\bar{\phi}_0(n-1; \lambda_1 + \lambda_2). \tag{27}$$

As sketched in I, the expression for the function $\bar{\phi}_0$ comes out from the Sturm-Liouville theory. When the second eigenvalue is made some two orders of magnitude smaller than the first one, a well-defined bistability (or bifurcation) in the stationary transition probability starts developing. This was shown in I for the simple case of an initial free random walk, i.e., $f_0(n) = b_0(n) = 1/2$, and will be shown here for the two-axes generalization (Figs. 1 and 2).

IV. UNCORRELATED TWO-AXES GENERALIZATION

The simplest generalization of the results obtained in I is to the case of homogeneous (i.e., free and/or anisotropic) RW's along two uncorrelated discrete axes.

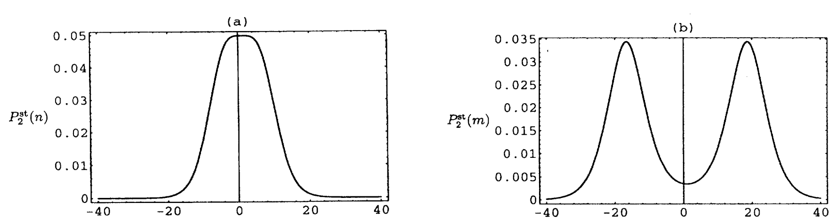

FIG. 1. Stationary states $P_2^{st}(n)$ and $P_2^{st}(m)$ for two uncorrelated free RW along two axes A and B with the two eigenvalues as follows: (a) $\lambda_{1A} = 0.01$ and $\lambda_{2A} = 0.01$; (b) $\lambda_{1B} = 0.01$ and $\lambda_{2B} = 0.0005$. This is a case with bistability along one axis only.

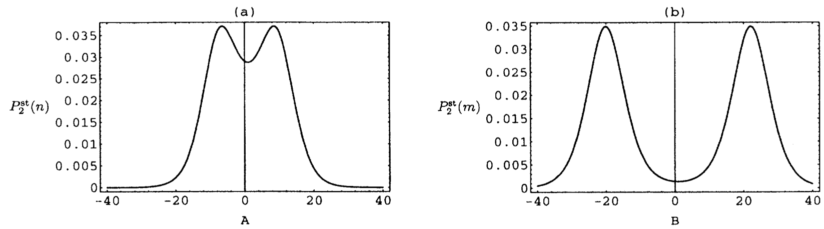

FIG. 2. The same as in Fig. 1 but with the following set of eigenvalues: (a) $\lambda_{1A} = 0.01$ and $\lambda_{2A} = 0.005$; (b) $\lambda_{1B} = 0.01$ and $\lambda_{2B} = 0.0002$. This case displays bistability along both axes.

This pair of axes can be anything like space and time, light front discrete coordinates, energy and space, two space axes, and so on. When the two axes are supposed uncorrelated, one can write down easily the matrix form for the independent RW's along the two axes, since this case corresponds to the separable 2D potentials in supersymmetric quantum mechanics. The algebra of the supercharge operators can be realized by writing $Q^- = q^- \times \sigma_+$ and $Q^+ = q^+ \times \sigma_-$, with $q^- = \begin{pmatrix} A^- & 0 \\ 0 & B^- \end{pmatrix}$ and $q^+ = \begin{pmatrix} A^+ & 0 \\ 0 & B^+ \end{pmatrix}$. The $\sigma_\pm$ matrices are again the Pauli matrices. The symbols A and B correspond to the first and second axis, respectively, and are given by expressions of the type 7(a) and 7(b) or 8(a) and 8(b) above. The total 4×4 "Hamiltonian" matrix can be written as

$$\mathcal{H}_{AB} = \begin{pmatrix} A^+A^- & 0 & 0 & 0 \\ 0 & B^+B^- & 0 & 0 \\ 0 & 0 & A^-A^+ & 0 \\ 0 & 0 & 0 & B^-B^+ \end{pmatrix}. \qquad (28)$$

The supersymmetric partner "Hamiltonians" are diagonalized 2×2 matrices, with each diagonal component depending on one axis alone.

Even this trivial two-axes generalization implies nevertheless a richer spectrum of possibilities for the physical situations, allowing us to have various combinations for the two stationary states $P_2^{st}(n)$ and $P_2^{st}(m)$ on the two axes depending on the choice of the two eigenvalues on each axis. We plotted for illustration a discrete free RW case with bistability on one axis alone and another one with bistability on both axes (see Figs. 1 and 2). The plotted functions are of the type obtained by Jauslin $P_2^{st} = \frac{2}{\phi_1(n)\phi_1(n-1)}$ with

$$\phi_1(n) = \left[\frac{1}{2a(n)}\right]^{1/2} \sinh(\gamma_2 n - \delta_2) - \left[\frac{a(n)}{2}\right]^{1/2} \sinh\left[\gamma_2(n-1) - \delta_2\right], \qquad (29)$$

where $a(n) = \frac{\cosh(\gamma_1 n - \delta_1)}{\cosh[\gamma_1(n-1)-\delta_1]}$, and $\gamma_1 = \text{arccosh}(1+\lambda_1)$, $\gamma_2 = \text{arccosh}(1+\lambda_1+\lambda_2)$, with λ_1 and λ_2 shifting constants, and δ_1 and δ_2 arbitrary constants.

For homogeneous RW's in the continuous limit the matrices $q^\pm$ can be written in terms of the "superpotentials" $W_A(n)$ and $W_B(m)$ on the two axes in the form

$$q^\pm = \begin{pmatrix} \pm\frac{\partial}{\partial n} + W_A(n) & 0 \\ 0 & \pm\frac{\partial}{\partial m} + W_B(m) \end{pmatrix} \qquad (30)$$

or

$$q^\pm = \begin{pmatrix} \pm\frac{\partial}{\partial x} + \sqrt{\Delta}\tanh[\sqrt{\Delta}(x-x_0)] & 0 \\ 0 & \pm\frac{\partial}{\partial y} + \sqrt{\Delta}\tanh[\sqrt{\Delta}(y-y_0)] \end{pmatrix}, \qquad (31)$$

where $\Delta = 2[1 - 2(c_1c_2)^{1/2} + \lambda]$, and we rescaled the coordinates $x = 2^{-1/2}(c_1c_2)^{-1/4}n$, $y = 2^{-1/2}(c_1c_2)^{-1/4}m$; x_0 and y_0 are integration constants. The explicit form of $W_{A,B}$ results from solving a Riccati equation of the type Eq. (19) with λ replaced by $[1 - 2(c_1c_2)^{1/2} + \lambda]$.

V. CONCLUSIONS

The simple supersymmetric algebraic schemes for RW's developed in this paper, as a continuation of I, may have many applications, both for the 1D case and even more for the two-axes case of the preceding section, either uncorrelated or correlated. It is known that the three-sites (or states) master equation applies to one-step processes [7] such as the number of molecules in a chemical species, the number of photons in a lasing mode, or the number of electrons on a capacitor. It will be of interest to apply supersymmetric algebraic schemes to multistep processes as well. It will be also very useful to find closed analytical forms for the more realistic correlated two-axes case, which must be studied carefully.

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